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Expectation and Transition Probability

Stanley P. Gudder

Department of Mathematics and Computer Science, University of Denver, Denver, Colorado 80208

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It is shown that a separating or order-determining set of states on a quantum logic need not determine the expectations of observables. A formula is derived for the transition probability between states. Using this formula, it is shown that the propositions do not determine the transition probability in a certain sense. The form of the transition probability is derived for pure states on Hilbert space, dominated normal states on a von Neumann algebra, and absolutely continuous states on a measurable space. A metric is defined in terms of the transition probability.

1. INTRODUCTION

It is part of the folklore of quantum mechanics that observables are determined by propositions. By this it is meant that complete information about the yes–no experiments for a quantum system determines complete information about the observables of the system. In a certain sense this may be true since any observable can be decomposed into propositions. For example, in certain algebraic approaches, a bounded observable is represented by a self-adjoint operator A in a von Neumann algebra $\mathcal{Q}$ and by the spectral theorem $A=\int\lambda P^A(d\lambda)$, where the spectral measure $P^A(E)\in\mathcal{Q}$ for all Borel sets $E\in B(\mathbb{R})$. As another example, in the quantum logic approach an observable is represented by a σ homomorphism from $B(\mathbb{R})$ into the set of propositions.

However, as we shall show, this bit of folklore is not true in another sense. If the probabilities that propositions are true are known for a large number of states, this does not determine the distributions of observables. More precisely, one can have a set of states which separate propositions in a strong way and yet do not separate the expectations of observables.

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Moreover, the transition probability between two states can be found using observables but not using propositions.

2. EXPECTATION DETERMINING

Let $(L, \leqslant)$ be a partially ordered set (poset) with least and greatest elements 0 and 1, respectively. We denote the greatest lower bound and least upper bound of $a, b \in L$, if they exist, by $a \wedge b$ and $a \vee b$, respectively. We say that L is *orthocomplemented* if there exists a map $' : L \to L$ such that $a'' = a$, $a \leqslant b$ implies that $b' \leqslant a'$, and $a \vee a' = 1$. Two elements $a, b \in L$ are *orthogonal* ($a \perp b$) if $a \leqslant b'$. We say that L is *σ-orthocomplete* if $\bigvee a_i$ exists for any sequence of mutually orthogonal elements $a_i \in L$. A σ-orthocomplete, orthocomplemented poset L is *orthomodular* if $a \leqslant b$ implies that $b = a \vee (b \wedge a')$. We call a σ-orthocomplete orthomodular poset a *logic* (Gudder, 1979; Jauch, 1968; Mackey, 1963; Piron, 1976; Varadarajan, 1968, 1970). The elements of L are called *propositions*.

A *state* on a logic L is a map α from L into the unit interval $[0,1] \subseteq \mathbb{R}$ such that $\alpha(1) = 1$ and $\alpha(\bigvee a_i) = \sum \alpha(a_i)$ whenever $a_i \perp a_j$, $i \neq j = 1, 2, \ldots$. A set of states S on L is *separating* if $\alpha(a) = \alpha(b)$ for every $\alpha \in S$ implies that $a = b$. We say that S is *order-determining* if $\alpha(a) \leqslant \alpha(b)$ for all $\alpha \in S$ implies that $a \leqslant b$. Finally, S is *strongly order-determining* if $\alpha(b) = 1$ whenever $\alpha(a) = 1$ implies that $a \leqslant b$. An *observable* is a map $x : B(\mathbb{R}) \to L$ such that $x(\mathbb{R}) = 1$, $E \cap F = \varnothing$ implies that $x(E) \perp x(F)$, and $x(\cup E_i) = \bigvee x(E_i)$ whenever $E_i \cap E_j = \varnothing$, $i \neq j = 1, 2, \ldots$.

If x is an observable and α a state, the *distribution* of x in the state α is the probability measure $\alpha_x(E) \equiv \alpha[x(E)]$ on $B(\mathbb{R})$. The *expectation* of x in the state α (if it exists) is $\alpha(x) = \int \lambda \alpha_x(d\lambda)$. A set of states S is *expectation determining* if $\alpha(x) = \alpha(y)$ (whenever one side exists) for every $\alpha \in S$ implies that $x = y$.

Let S be a separating set of states on L. Is S expectation determining? The following simple example shows that the answer, in general, is no.

Example 1. Let L be the Boolean algebra of all subsets of the set $\{1,2\}$, and let α be the state defined by $\alpha(\varnothing) = 0$, $\alpha(\{1,2\}) = 1$, $\alpha(\{1\}) = 1/3$, and $\alpha(\{2\}) = 2/3$. Let x and y be the observables defined by $x(\{3\}) = \{1\}$, $x(\{6\}) = \{2\}$, and $y(\{4\}) = \{1\}$, $y(\{11/2\}) = \{2\}$. It is clear that $\{\alpha\}$ is a separating set of states. However,

$$\alpha(x) = 3(1/3) + 6(2/3) = 5$$

$$\alpha(y) = 4(1/3) + (11/2)(2/3) = 5$$

so $\{\alpha\}$ is not expectation determining.

Let S be an order-determining set of states on L. Is S expectation determining in general? Again, the answer is no, as the following example shows. (Since order-determining implies separating, the following example supersedes Example 1. However, it is much more complicated.)

Example 2. Using the usual set-theoretic order and complementation, let L be the logic of subsets of $\{1,2,\ldots,9\}$ generated by the sets $A=\{1,2,3\}$, $B=\{4,5,6\}$, $C=\{7,8,9\}$, $D=\{1,4,7\}$, $E=\{2,5,8\}$, $F=\{3,6,9\}$. Let $\alpha_1, \alpha_2, \alpha_3$ be the states that are uniquely defined by the following table:

	A	B	C	D	E	F
α_1	1/8	2/8	5/8	1/10	0	9/10
α_2	2/8	5/8	1/8	0	9/10	1/10
α_3	5/8	1/8	2/8	9/10	1/10	0

It is not hard to check that $\{\alpha_1, \alpha_2, \alpha_3\}$ is an order-determining set. Let x and y be the observables defined by $x(\{0\})=A\cup B$, $x(\{8\})=C$, $y(\{158/73\})=D$, $y(\{38/73\})=E$, $y(\{388/73\})=F$. Then

$$\alpha_1(x)=0(3/8)+8(5/8)=5$$

$$\alpha_2(x)=0(7/8)+8(1/8)=1$$

$$\alpha_3(x)=0(6/8)+8(2/8)=2$$

$$\alpha_1(y)=(158/73)(1/10)+(38/73)(0)+(388/73)(9/10)=5$$

$$\alpha_2(y)=(158/73)(0)+(38/73)(9/10)+(388/73)(1/10)=1$$

$$\alpha_3(y)=(158/73)(9/10)+(38/73)(1/10)+(388/73)(0)=2$$

Hence, $\{\alpha_1, \alpha_2, \alpha_3\}$ is not expectation determining.

If S is strongly order-determining it is shown in Gudder (1966) that S is expectation determining for x and y if one of the two observables has a spectrum with at most one limit point. In general, it is unknown if a strongly order-determining set of states is expectation determining.

3. TRANSITION PROBABILITY

Let α, β be states and let x be an observable on the logic L. It is not hard to show that there exists a finite Borel measure σ such that $\alpha_x, \beta_x \ll \sigma$

and that the expression

$$T_x^{1/2}(\alpha,\beta)=\int\left(\frac{d\alpha_x}{d\sigma}\right)^{1/2}\left(\frac{d\beta_x}{d\sigma}\right)^{1/2}d\sigma$$

is independent of σ. Following Cantoni (1975), we define the *transition probability* of α to β by

$$T(\alpha,\beta)=\inf T_x(\alpha,\beta)$$

where the infimum is taken over all observables. For a physical motivation of this definition see Gudder (1978).

If $a\in L$, we define $\hat{a}$ to be the unique observable satisfying $\hat{a}(\{1\})=a$, $\hat{a}(\{0\})=a'$. Do the propositions determine the transition probability in the following sense?

$$T(\alpha,\beta)=\inf_{a\in L} T_{\hat{a}}(\alpha,\beta)$$

In this section we shall show that the answer, in general, is no. To do this, we first obtain a useful formula for $T(\alpha,\beta)$. As usual, a *Borel partition* of $\mathbb{R}$ is a sequence of mutually disjoint sets $E_i\in B(\mathbb{R})$ such that $\mathbb{R}=\cup E_i$. A *maximal orthogonal sequence* in L is a sequence of mutually orthogonal propositions a_i such that $\vee a_i=1$.

Theorem 1. $T^{1/2}(\alpha,\beta)=\inf\Sigma\alpha(a_i)^{1/2}\beta(a_i)^{1/2}$, where the infimum is over all finite maximal orthogonal sequences in L.

Proof. Let E_i be a finite Borel partition of $\mathbb{R}$. Applying Schwarz's inequality we have

$$T_x^{1/2}(\alpha,\beta)=\int_{\cup E_i}\left(\frac{d\alpha_x}{d\sigma}\right)^{1/2}\left(\frac{d\beta_x}{d\sigma}\right)^{1/2}d\sigma=\sum\int_{E_i}\left(\frac{d\alpha_x}{d\sigma}\right)^{1/2}\left(\frac{d\beta_x}{d\sigma}\right)^{1/2}d\sigma$$

$$\leqslant\sum\left(\int_{E_i}\frac{d\alpha_x}{d\sigma}d\sigma\right)^{1/2}\left(\int_{E_i}\frac{d\beta_x}{d\sigma}d\sigma\right)^{1/2}=\sum\alpha_x(E_i)^{1/2}\beta_x(E_i)^{1/2}$$

Hence,

$$T_x^{1/2}(\alpha,\beta)\leqslant\inf\sum\alpha_x(E_i)^{1/2}\beta_x(E_i)^{1/2}$$

where the infimum is over all finite Borel partitions. Let $\varepsilon>0$ be given and let x be an observable. If $\sigma=\alpha_x+\beta_x$, then $\alpha_x(E)$, $\beta_x(E)\leqslant\sigma(E)$ for every

$E \in B(R)$, so we can assume that $d\alpha_x/d\sigma$, $d\beta_x/d\sigma \leqslant 1$ and $\sigma(\mathbb{R}) \leqslant 2$. Now assume there exists a $c > 0$ such that $d\alpha_x/d\sigma$, $d\beta_x/d\sigma \geqslant c$. Then there exist a Borel partition E_i and constants $c \leqslant c_i$, $d_i \leqslant 1$, $1 \leqslant i \leqslant n$ such that

$$\left|\frac{d\alpha_x}{d\sigma} - \sum c_i \chi_{E_i}\right| \leqslant \varepsilon, \quad \left|\frac{d\beta_x}{d\sigma} - \sum d_i \chi_{E_i}\right| \leqslant \varepsilon$$

Then

$$\begin{aligned}
&\left|\frac{d\alpha_x}{d\sigma}\frac{d\beta_x}{d\sigma} - \sum c_i d_i \chi_{E_i}\right| \\
&\qquad \leqslant \left|\frac{d\alpha_x}{d\sigma}\frac{d\beta_x}{d\sigma} - \frac{d\alpha_x}{d\sigma}\sum d_i \chi_{E_i}\right| + \left|\frac{d\alpha_x}{d\sigma}\sum d_i \chi_{E_i} - \sum c_i d_i \chi_{E_i}\right| \\
&\qquad \leqslant \left|\frac{d\beta_x}{d\sigma} - \sum d_i \chi_{E_i}\right| + \left|\sum d_i \chi_{E_i}\left(\frac{d\alpha_x}{d\sigma} - \sum c_i \chi_{E_i}\right)\right| \\
&\qquad \leqslant \varepsilon + \varepsilon(1+\varepsilon)
\end{aligned}$$

Hence

$$\begin{aligned}
&\left|\left(\frac{d\alpha_x}{d\sigma}\right)^{1/2}\left(\frac{d\beta_x}{d\sigma}\right)^{1/2} - \sum c_i^{1/2} d_i^{1/2} \chi_{E_i}\right| \\
&\quad = \left|\frac{d\alpha_x}{d\sigma}\frac{d\beta_x}{d\sigma} - \sum c_i d_i \chi_{E_i}\right| \Big/ \left[\left(\frac{d\alpha_x}{d\sigma}\right)^{1/2}\left(\frac{d\beta_x}{d\sigma}\right)^{1/2} + \sum c_i^{1/2} d_i^{1/2} \chi_{E_i}\right] \\
&\quad \leqslant c^{-1}[\varepsilon + \varepsilon(1+\varepsilon)]
\end{aligned}$$

Also,

$$\begin{aligned}
|c_j \sigma(E_j) - \alpha_x(E_j)| &= \left|\int_{E_j} \sum c_i \chi_{E_i}\, d\sigma - \int_{E_j} \frac{d\alpha_x}{d\sigma}\, d\sigma\right| \\
&\leqslant \int_{E_j} \left|\sum c_i \chi_{E_i} - \frac{d\alpha_x}{d\sigma}\right| d\sigma \leqslant \varepsilon \sigma(E_j)
\end{aligned}$$

and

$$\begin{aligned}
|c_j^{1/2}\sigma(E_j)^{1/2} - \alpha_x(E_j)^{1/2}| &= |c_j \sigma(E_j) - \alpha_x(E_j)| \Big/ \left[c_j^{1/2}\alpha(E_j)^{1/2} + \alpha_x(E_j)^{1/2}\right] \\
&\leqslant c^{-1/2}\varepsilon \sigma(E_j)^{1/2}
\end{aligned}$$

Similarly,

$$|d_j^{1/2}\sigma(E_j)^{1/2}-\beta_x(E_j)^{1/2}|\leqslant c^{-1/2}\varepsilon\sigma(E_j)^{1/2}$$

Hence,

$$\left|\int\left(\frac{d\alpha_x}{d\sigma}\right)^{1/2}\left(\frac{d\beta_x}{d\sigma}\right)^{1/2}d\sigma-\sum\alpha_x(E_i)^{1/2}\beta_x(E_i)^{1/2}\right|$$

$$\leqslant\left|\int\left(\frac{d\alpha_x}{d\sigma}\right)^{1/2}\left(\frac{d\beta_x}{d\sigma}\right)^{1/2}d\sigma-\sum c_i^{1/2}d_i^{1/2}\sigma(E_i)\right|$$

$$+\left|\sum c_i^{1/2}d_i^{1/2}\sigma(E_i)-\sum\alpha_x(E_i)^{1/2}\beta_x(E_i)^{1/2}\right|$$

$$\leqslant\int\left|\left(\frac{d\alpha_x}{d\sigma}\right)^{1/2}\left(\frac{d\beta_x}{d\sigma}\right)^{1/2}-\sum c_i^{1/2}d_i^{1/2}\chi_{E_i}\right|d\sigma$$

$$+\left|\sum c_i^{1/2}d_i^{1/2}\sigma(E_i)-\sum c_i^{1/2}\sigma(E_i)^{1/2}\beta_x(E_i)^{1/2}\right|$$

$$+\left|\sum c_i^{1/2}\sigma(E_i)^{1/2}\beta_x(E_i)^{1/2}-\sum\alpha_x(E_i)^{1/2}\beta_x(E_i)^{1/2}\right|$$

$$\leqslant 2c^{-1}[\varepsilon+\varepsilon(1+\varepsilon)]+\sum c_i^{1/2}\sigma(E_i)^{1/2}|d_i^{1/2}\sigma(E_i)^{1/2}-\beta_x(E_i)^{1/2}|$$

$$+\sum\beta_x(E_I)^{1/2}|c_i^{1/2}\sigma(E_i)^{1/2}-\alpha_x(E_i)^{1/2}|$$

$$\leqslant 2c^{-1}[\varepsilon+\varepsilon(1+\varepsilon)]+\varepsilon c^{-1/2}\sum c_i^{1/2}\sigma(E_i)$$

$$+\varepsilon c^{-1/2}\sum\sigma(E_i)^{1/2}\left[d_i^{1/2}\sigma(E_i)^{1/2}+c^{-1/2}\varepsilon\sigma(E_i)^{1/2}\right]$$

$$\leqslant 2c^{-1}[\varepsilon+\varepsilon(1+\varepsilon)]+3c^{-1/2}\varepsilon+c^{-1}\varepsilon^2$$

Adjusting the constants, we conclude that

$$\sum\alpha_x(E_i)^{1/2}\beta_x(E_i)^{1/2}\leqslant\int\left(\frac{d\alpha_x}{d\sigma}\right)^{1/2}\left(\frac{d\beta_x}{d\sigma}\right)^{1/2}d\sigma+\varepsilon \tag{1}$$

Now suppose that $d\alpha_x/d\sigma$ or $d\beta_x/d\sigma$ is not bounded below by a positive constant. Then given $\varepsilon>0$, we have $d\alpha_x/d\sigma+\varepsilon$, $d\beta_x/d\sigma+\varepsilon\geqslant\varepsilon$. By the above argument, there is a finite Borel partition E_i such that

$$\sum \alpha_x(E_i)^{1/2}\beta_x(E_i)^{1/2} \leqslant \sum \left[\alpha_x(E_i)+\varepsilon\sigma(E_i)\right]^{1/2}\left[\beta_x(E_i)+\varepsilon\sigma(E_i)\right]^{1/2}$$

$$\leqslant \int \left(\frac{d\alpha_x}{d\sigma}+\varepsilon\right)^{1/2}\left(\frac{d\beta_x}{d\sigma}+\varepsilon\right)^{1/2} d\sigma+\varepsilon$$

$$\leqslant \int \left(\frac{d\alpha_x}{d\sigma}\frac{d\beta_x}{d\sigma}+2\varepsilon+\varepsilon^2\right)^{1/2} d\sigma+\varepsilon$$

$$\leqslant \int \left[\left(\frac{d\alpha_x}{d\sigma}\right)^{1/2}\left(\frac{d\beta_x}{d\sigma}\right)^{1/2}+(2\varepsilon+\varepsilon^2)^{1/2}\right] d\sigma+\varepsilon$$

$$\leqslant \int \left(\frac{d\alpha_x}{d\sigma}\right)^{1/2}\left(\frac{d\beta_x}{d\sigma}\right)^{1/2} d\sigma+2(2\varepsilon+\varepsilon^2)^{1/2}+\varepsilon$$

Adjusting constants again we conclude that (1) holds in general. It follows that

$$\inf \sum \alpha_x(E_i)^{1/2}\beta_x(E_i)^{1/2} \leqslant T_x^{1/2}(\alpha,\beta)+\varepsilon$$

Since $\varepsilon>0$ is arbitrary, we have

$$T_x^{1/2}(\alpha,\beta)=\inf \sum \alpha_x(E_i)^{1/2}\beta_x(E_i)^{1/2}$$

Hence,

$$T^{1/2}(\alpha,\beta)=\inf \sum \alpha_x(E_i)^{1/2}\beta_x(E_i)^{1/2} \tag{2}$$

where the infimum is over all observables and all finite Borel partitions. If x is an observable and E_i is a finite Borel partition, then $a_i=x(E_i)$ is a maximal orthogonal sequence. Conversely, if a_i is a maximal orthogonal sequence, then there exists an observable x and a Borel partition E_i such that $x(E_i)=a$. The result now follows from (2). ■

Using Theorem 1, our question about propositions determining the transition probability reduces to whether

$$T^{1/2}(\alpha,\beta)=\inf_{a\in L}\left[\alpha(a)^{1/2}\beta(a)^{1/2}+\alpha(a')^{1/2}\beta(a')^{1/2}\right]$$

Example 3. Let A, B, C, D, E, F be as in Example 2 and let $\alpha=\alpha_1$, $\beta=\alpha_2$ in Example 2. Then

$$T_A^{1/2} \equiv \alpha(A)^{1/2}\beta(A)^{1/2} + [1-\alpha(A)]^{1/2}[1-\beta(A)]^{1/2}$$

$$=(1/8)^{1/2}(2/8)^{1/2}+(7/8)^{1/2}(6/8)^{1/2}=(2^{1/2}+42^{1/2})/8$$

Similarly,

$$T_B^{1/2}=(10^{1/2}+18^{1/2})/8, \qquad T_C^{1/2}=(5^{1/2}+21^{1/2})/8$$

$$T_D^{1/2}=3/10^{1/2}, \qquad T_E=1/10^{1/2}, \qquad T_F=3/5$$

Hence,

$$\inf_a \left[\alpha(a)^{1/2}\beta(a)^{1/2}+\alpha(a')^{1/2}\beta(a')^{1/2}\right]=1/10^{1/2}$$

Besides the two-element maximal orthogonal sequences, there are two three-element maximal orthogonal sequences $\{A, B, C\}$ and $\{D, E, F\}$. Now

$$\alpha(A)^{1/2}\beta(A)^{1/2}+\alpha(B)^{1/2}\beta(B)^{1/2}+\alpha(C)^{1/2}\beta(C)^{1/2}$$

$$=(1/8)^{1/2}(2/8)^{1/2}+(2/8)^{1/2}(5/8)^{1/2}+(5/8)^{1/2}(1/8)^{1/2}$$

$$=(2^{1/2}+5^{1/2}+10^{1/2})/8$$

And

$$\alpha(D)^{1/2}\beta(D)^{1/2}+\alpha(E)^{1/2}\beta(E)^{1/2}+\alpha(F)^{1/2}\beta(E)^{1/2}=3/10$$

Hence,

$$T^{1/2}(\alpha,\beta)=3/10<1/10^{1/2}=\inf_a\left[\alpha(a)^{1/2}\beta(a)^{1/2}+\alpha(a')^{1/2}\beta(a')^{1/2}\right]$$

One can check that the states in Example 3 are not pure. If α and β are pure, a more complicated example is needed.

Example 4. Under the usual set-theoretic order and complementation, let L be the logic of all subsets of $\Omega=\{1,2,3,4,5,6\}$ with an even number of elements. Define a state α on L by $\alpha(\varnothing)=0$, $\alpha(\Omega)=1$,

$$\alpha(\{i,j\})=\begin{cases} 1/2 & \text{if } i,j\neq 1 \\ 0 & \text{if } i=1 \text{ or } j=1 \end{cases}$$

$$\alpha(\{i,j,k,l\})=\begin{cases} 1 & \text{if } i,j,k,l\neq 1 \\ 1/2 & \text{if } i \text{ or } j \text{ or } k \text{ or } l=1 \end{cases}$$

It is not hard to show that α is a state. Also, α is the unique state satisfying $\alpha(\{i,j\})=0$ if $i=1$ or $j=1$. Indeed, suppose β is a state satisfying $\beta(\{i,j\})=0$ if $i=1$ or $j=1$. Then

$$\beta(\{6,2\})=\lambda\rightarrow\beta(\{3,4\})=1-\lambda\rightarrow\beta(\{5,2\})$$
$$=\lambda\rightarrow\beta(\{6,3\})=1-\lambda\rightarrow\beta(\{4,5\})=\lambda$$

But, $\beta(\{6,2\})+\beta(\{4,5\})=1$ and hence $\lambda=1/2$. It follows that $\beta=\alpha$. The state α is pure since $\alpha=\lambda\alpha_1+(1-\lambda)\alpha_2$ for $0<\lambda<1$ implies that if $i=1$ or $j=1$, then

$$0=\alpha(\{i,j\})=\lambda\alpha_1(\{i,j\})+(1-\lambda)\alpha_2(\{i,j\})$$

Hence, $\alpha_1(\{i,j\})=\alpha_2(\{i,j\})=0$. Since α is the unique state with this property, $\alpha_1=\alpha_2$ so α is pure.

Define a state β on L just like α above with 1 replaced by 2. Then β is also pure. By considering the different cases one can show that

$$\inf_{A\in L}\left[\alpha(A)^{1/2}\beta(A)^{1/2}+\alpha(A')^{1/2}\beta(A')^{1/2}\right]=2^{-1/2}$$

But

$$\alpha(\{1,3\})^{1/2}\beta(\{1,3\})^{1/2}+\alpha(\{2,4\})^{1/2}\beta(\{2,4\})^{1/2}$$
$$+\alpha(\{5,6\})^{1/2}\beta(\{5,6\})^{1/2}=1/2<2^{-1/2}.$$

In fact, one can show that $T^{1/2}(\alpha,\beta)=1/2$.[1]

[1]The author is indebted to professor Richard Greechie for pointing out the above example.

4. PROPERTIES OF THE TRANSITION PROBABILITY

In this section we shall use our equation

$$T^{1/2}(\alpha,\beta)=\inf\sum\alpha(a_i)^{1/2}\beta(a_i)^{1/2} \tag{3}$$

which we have proved in Theorem 1 to derive some properties of the transition probability. In particular, we shall derive some formulas for $T(\alpha,\beta)$ in important special cases.

The following lemma due to Cantoni (1975) is easily proved using (3).

Lemma 2. (a) $T(\alpha,\beta)=T(\beta,\alpha)$, (b) $0\leqslant T(\alpha,\beta)\leqslant 1$, (c) $T(\alpha,\beta)=1$ if and only if $\alpha=\beta$.

Proof. (a) is trivial and (b) follows from Schwarz's inequality. (c) Clearly, $T(\alpha,\alpha)=1$. Now suppose that $T(\alpha,\beta)=1$, and let $a\in L$. Applying (3) in the Hilbert space $\mathbb{R}^2$ we have

$$\alpha(a)^{1/2}\beta(a)^{1/2}+[1-\alpha(a)]^{1/2}[1-\beta(a)]^{1/2}$$
$$=1=\left\|\left(\alpha(a)^{1/2},[1-\alpha(a)]^{1/2}\right)\right\|\left\|\left(\beta(a)^{1/2},[1-\beta(a)]^{1/2}\right)\right\|$$

We thus have equality in Schwarz's inequality and hence there exists a $c\in\mathbb{R}$ such that $\alpha(a)=c\beta(a)$ and $1-\alpha(a)=c[1-\beta(a)]$. Hence, $c=1$ and $\alpha(a)=\beta(a)$. Therefore, $\alpha=\beta$. ■

Let $L(H)$ be the logic of all orthogonal projections on a Hilbert space H. For a pure state α on L we let $\hat{\alpha}$ be a representing unit vector. We now obtain a simple proof of a result due to Cantoni (1975). (It should be noted that Cantoni's result applies to a slightly more general context.)

Theorem 3. If α and β are pure states on $L(H)$, then $T(\alpha,\beta)=|\langle\alpha,\beta\rangle|^2$.

Proof. Let P_i be a maximal orthogonal sequence in $L(H)$ and let P_α be the one-dimensional projection onto $\hat{\alpha}$. Then

$$\alpha(P_\alpha)^{1/2}\beta(P_\alpha)^{1/2}+\alpha(P_\alpha')^{1/2}\beta(P_\alpha')^{1/2}$$
$$=\langle P_\alpha\hat{\alpha},\hat{\alpha}\rangle^{1/2}\langle P_\alpha\hat{\beta},\hat{\beta}\rangle^{1/2}+\langle P_\alpha'\hat{\alpha},\hat{\alpha}\rangle^{1/2}\langle P_\alpha'\hat{\beta},\hat{\beta}\rangle^{1/2}$$
$$=|\langle\hat{\alpha},\hat{\beta}\rangle|=\left|\left\langle\sum P_i\hat{\alpha},\hat{\beta}\right\rangle\right|\leqslant\sum|\langle P_i\hat{\alpha},\hat{\beta}\rangle|$$
$$\leqslant\sum\|P_i\hat{\alpha}\|\,\|P_i\hat{\beta}\|=\sum\langle P_i\hat{\alpha},\hat{\alpha}\rangle^{1/2}\langle P_i\hat{\beta},\hat{\beta}\rangle^{1/2}=\sum\alpha(P_i)^{1/2}\beta(P_i)^{1/2}$$

It follows that

$$|\langle\hat{\alpha},\hat{\beta}\rangle|=\inf\sum\alpha(P_i)^{1/2}\beta(P_i)^{1/2}=T^{1/2}(\alpha,\beta).$$ ■

Let $L(\mathcal{Q})$ be the logic of all orthogonal projections in a von Neumann algebra $\mathcal{Q}$, and let α be a faithful normal state on $\mathcal{Q}$. If β is a normal state on $\mathcal{Q}$ such that $\beta(A^*A) \leqslant K\alpha(A^*A)$ for every $A \in \mathcal{Q}$ and some $K>0$, then there exists a unique positive operator $T \in \mathcal{Q}$ such that $\beta(A)=\alpha(TAT)$ for all $A \in \mathcal{Q}$ (Sakai, 1965). The next result gives the "relative entropy" studied in Benoist and Marchand (1979), Benoist et al. (1979), Gudder et al. (1979), and Marchand and Wyss (1977).

Theorem 4. $T(\alpha,\beta)=[\alpha(T)]^2$.

Proof. Let P_i be a finite maximal orthogonal sequence in $L(\mathcal{Q})$. Then applying Schwarz's inequality we have

$$\alpha(T)=\alpha\left(\sum P_iT\right)=\sum \alpha(P_iT)=\sum \alpha[P_i(P_iT)]$$

$$\leqslant \sum \alpha(P_i)^{1/2}\alpha(TP_iT)^{1/2}=\sum \alpha(P_i)^{1/2}\beta(P_i)^{1/2}$$

Hence, $\alpha(T) \leqslant T^{1/2}(\alpha,\beta)$. Now suppose that T is invertible and let $\varepsilon>0$. By the spectral theorem, there exists a $c>0$ and $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$ such that $c \leqslant \lambda_i \leqslant \|T\|$, $1 \leqslant i \leqslant n$, and a maximal orthogonal sequence $P_1,\ldots,P_n \in L(\mathcal{Q})$ such that $\Sigma\lambda_iP_i \leqslant T$, $P_iT=TP_i$ for $1 \leqslant i \leqslant n$, and $\|T-\Sigma\lambda_iP_i\| \leqslant \varepsilon$. Then

$$\left\|T^2-\sum \lambda_1^2P_i\right\|=\left\|\left(T-\sum \lambda_iP_i\right)\left(T+\sum \lambda_iP_i\right)\right\|$$

$$\leqslant \varepsilon\left[\|T\|+\left\|\sum \lambda_iP_i\right\|\right] \leqslant \varepsilon(2\|T\|+\varepsilon)$$

Since $\Sigma\lambda_i^2P_i \leqslant T$ we have

$$\left|\alpha(T)-\sum \alpha(P_i)^{1/2}\alpha(TP_iT)^{1/2}\right|$$

$$\leqslant\left|\alpha(T)-\sum \lambda_i\alpha(P_i)\right|+\left|\sum \lambda_i\alpha(P_i)-\sum \alpha(P_i)^{1/2}\alpha\left(T^2P_i\right)^{1/2}\right|$$

$$\leqslant\left|\alpha\left(T-\sum \lambda_iP_i\right)\right|+\left|\sum \alpha(P_i)^{1/2}\left[\lambda_i\alpha(P_i)^{1/2}-\alpha\left(T^2P_i\right)^{1/2}\right]\right|$$

$$\leqslant\left\|T-\sum \lambda_iP_i\right\|+\sum \alpha(P)^{1/2}\left[\alpha\left(T^2P_i\right)-\lambda_i^2\alpha(P_i)\right]$$

$$\times\left[\lambda_i\alpha(P_i)^{1/2}+\alpha\left(T^2P_i\right)^{1/2}\right]^{-1}$$

$$\leqslant \varepsilon+\sum \lambda_i^{-1}\left[\alpha\left(T^2P_i\right)-\lambda_i^2\alpha(P_i)\right]$$

$$\leqslant \varepsilon+c^{-1}\alpha\left[\sum\left(T^2P_i-\lambda_i^2P_i\right)\right]=\varepsilon+c^{-1}\alpha\left(T^2-\sum \lambda_i^2P_i\right)$$

$$\leqslant \varepsilon+c^{-1}\left\|T^2-\sum \lambda_i^2P_i\right\| \leqslant \varepsilon+c^{-1}\varepsilon(2\|T\|+\varepsilon)$$

Adjusting the constants, we conclude that if T is invertible and $\varepsilon>0$ we have

$$\left|\alpha(T)-\sum\alpha(P_i)^{1/2}\alpha(TP_iT)^{1/2}\right|\leqslant\varepsilon \tag{4}$$

Now if T is not invertible, then $\varepsilon I+T$ is invertible and from (4) it follows that there exists a finite maximal orthogonal sequence $P_i\in L(\mathfrak{A})$ such that

$$\sum\alpha(P_i)^{1/2}\alpha\left[(\varepsilon I+T)P_i(\varepsilon I+T)\right]^{1/2}<\alpha(\varepsilon I+T)+\varepsilon$$

Hence,

$$\alpha(T)+2\varepsilon\geqslant\sum\alpha(P_i)^{1/2}\alpha\left[(\varepsilon^2I+2\varepsilon T+T^2)P_i\right]^{1/2}$$

$$\geqslant\sum\alpha(P_i)^{1/2}\alpha\left(T^2P_i\right)^{1/2}=\sum\alpha(P_i)^{1/2}\beta(P_i)^{1/2}$$

Thus, $T^{1/2}(\alpha,\beta)\leqslant\alpha(T)+2\varepsilon$ and since $\varepsilon>0$ is arbitrary, we have $T^{1/2}(\alpha,\beta)\leqslant\alpha(T)$. ■

As another application, let (Ω,Σ) be a measurable space. Then the σ-algebra Σ is a logic. If α and β are states on Σ, then α and β are probability measures. Suppose $\beta\ll\alpha$. Then using techniques similar to those in Theorems 1 and 4 we can obtain the following result.

Theorem 5. $T(\alpha,\beta)=\left[\int\left(\frac{d\beta}{d\alpha}\right)^{1/2}d\alpha\right]^2$

In the Hilbertian logic $L(H)$, it is easy to show that $\|\hat{\alpha}-\hat{\beta}\|=\{2[1-T^{1/2}(\alpha,\beta)]\}^{1/2}$. Motivated by this, if α and β are states on a logic L we define a distance function $d(\alpha,\beta)=\{2[1-T^{1/2}(\alpha,\beta)]\}^{1/2}$.

Theorem 6. $d(\alpha,\beta)$ is a metric on the states of a logic L.

Proof. It is clear that $d(\alpha,\beta)=d(\beta,\alpha)$, $d(\alpha,\beta)\geqslant0$ and $d(\alpha,\alpha)=0$ for all states α,β on L. We now prove the triangle inequality. If x,y,z are unit vectors of a real inner product space, then

$$\|x-y\|\leqslant\|x-z\|+\|z-y\|$$

In terms of the inner product this becomes

$$[2-2\langle x,y\rangle]^{1/2}\leqslant[2-2\langle x,z\rangle]^{1/2}+[2-2\langle z,y\rangle]^{1/2} \tag{5}$$

Let a_i be a finite maximal orthogonal sequence in L and let α be a state. Since $\Sigma\alpha(a_i)=1$, the sequence $\alpha(a_i)^{1/2}$ is a unit vector in l_2. If α, β, γ are states, then from (5) we have

$$\left[2-2\sum\alpha(a_i)^{1/2}\beta(a_i)^{1/2}\right]^{1/2}\leqslant\left[2-2\sum\alpha(a_i)^{1/2}\gamma(a_i)^{1/2}\right]^{1/2}+\left[2-2\sum\gamma(a_i)^{1/2}\beta(a_i)^{1/2}\right]^{1/2}$$

Hence,

$$\begin{aligned}d(\alpha,\beta)&=\left[2-2\inf\sum\alpha(a_i)^{1/2}\beta(a_i)^{1/2}\right]^{1/2}\\&=\sup\left[2-2\sum\alpha(a_i)^{1/2}\beta(a_i)^{1/2}\right]^{1/2}\\&\leqslant\sup\left[2-2\sum\alpha(a_i)^{1/2}\gamma(a_i)^{1/2}\right]^{1/2}\\&+\sup\left[2-2\sum\alpha(a_i)^{1/2}\beta(a_i)^{1/2}\right]^{1/2}\\&=d(\alpha,\gamma)+d(\gamma,\beta).\end{aligned}$$

■

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